Neural Networks II: Backpropagation

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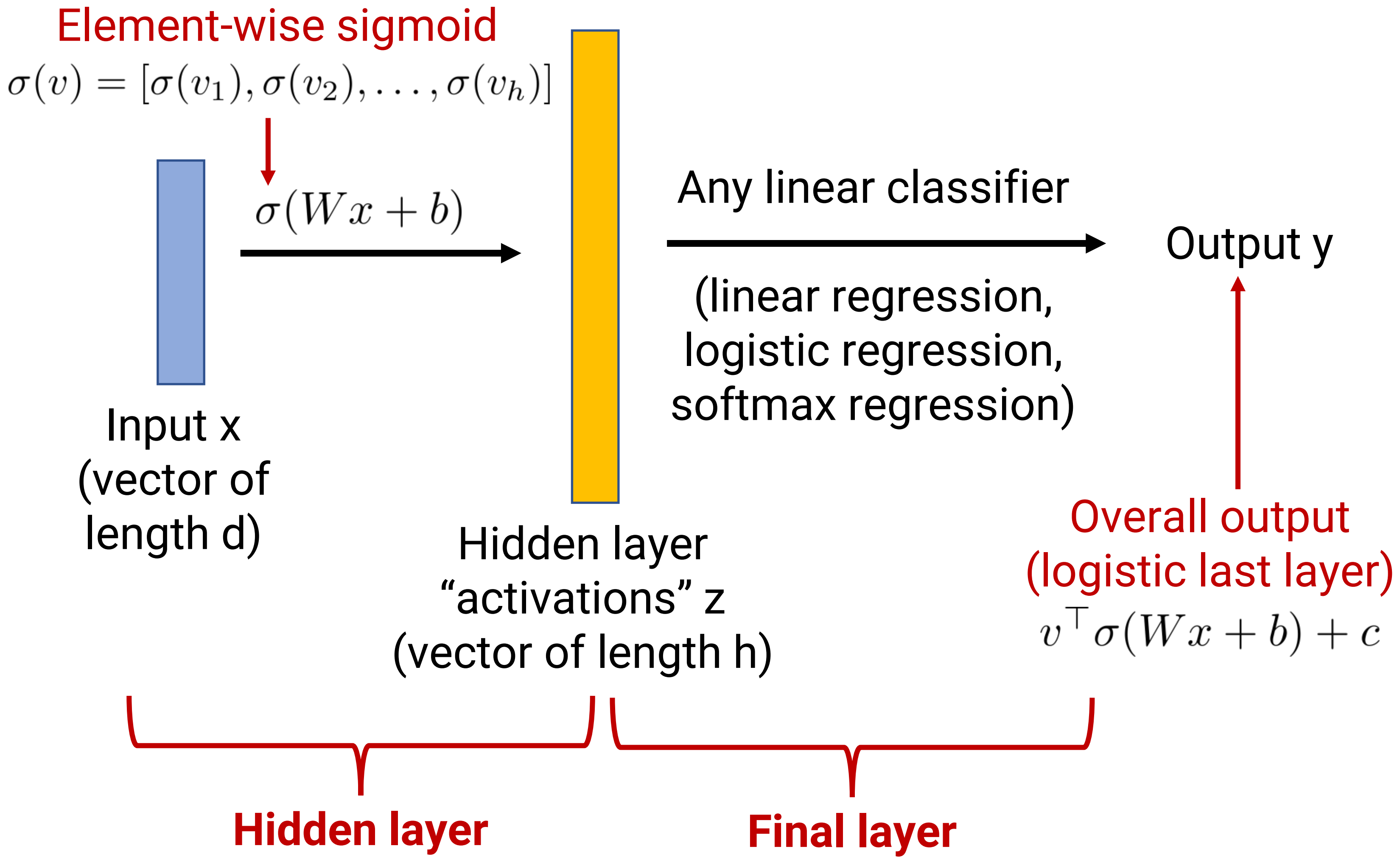
USC CSCI 467, Fall 2023
September 26, 2023

Recap: Neural Networks



- Hidden layer: A bunch of logistic regression classifiers
  - Parameters: w_j and b_j for each classifier
  - Equivalently: matrix W (h x d) and vector b (length h)
  - Produces "activations" = learned feature vector
- Final layer: A linear classifier
  - E.g. if logistic regression, has parameter vector *v* and bias *c*
- Parameters of model are *θ = (W, b, v, c)*
- **One neural network is built of many basic mathematical operations put together**
  - Sum, product, exponentiation, etc.

Recap: Stochastic gradient descent

Goal: Find parameters θ that minimize $\frac{1}{n}\sum_{i=1}^{n}\ell\left(y^{(i)}, g(x^{(i)})\right)$

Model's output, depends on parameters θ

Gradient Descent

$$\theta \leftarrow \theta - \eta \cdot \underbrace{\frac{1}{n}\sum_{i=1}^{n}\nabla_\theta \ell\left(y^{(i)}, g(x^{(i)})\right)}_{\text{Average of per-example gradients}}$$

- Intuition: Tweaking θ in direction of negative gradient should decrease loss
- Disadvantage: 1 update is O(n) time
- Idea: Approximate with sample mean

Stochastic Gradient Descent

1. Sample a *batch* B of examples from the training dataset
2. Do the update

$$\theta \leftarrow \theta - \eta \cdot \underbrace{\frac{1}{|B|}\sum_{(x,y)\in B}\nabla_\theta \ell\left(y, g(x)\right)}_{\text{Sample mean within batch}}$$

Today: How to compute gradients?

$$\theta \leftarrow \theta - \eta \cdot \frac{1}{|B|} \sum_{(x,y)\in B} \underbrace{\nabla_\theta \ell\left(y, g(x)\right)}_{\text{How to compute this gradient?}}$$

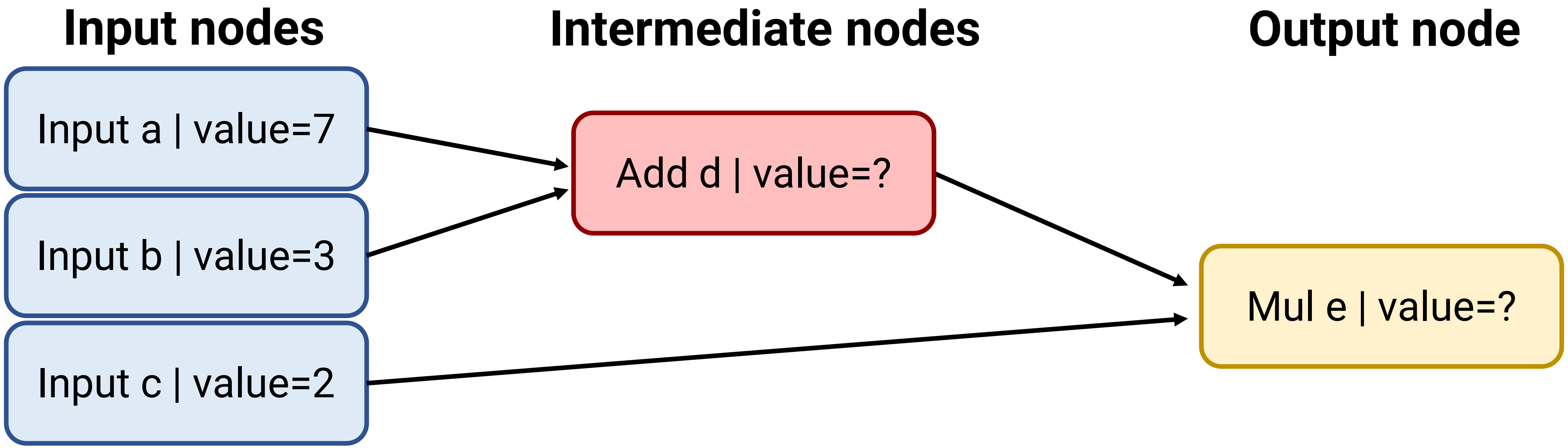
- Taking gradients by hand is tedious, and we have to re-do it every time we want to change the "architecture"
  - Change number of layers
  - Change from sigmoid to tanh
  - Etc…
- Let's make a program do it for us!

Today's Plan

- The computation graph
- Backpropagation on trees
- Backpropagation on DAGs

Computation Graph

<u>Computation graph for ***(a + b) * c*** when *a=7, b=3, c=2*</u>



Computation Graph

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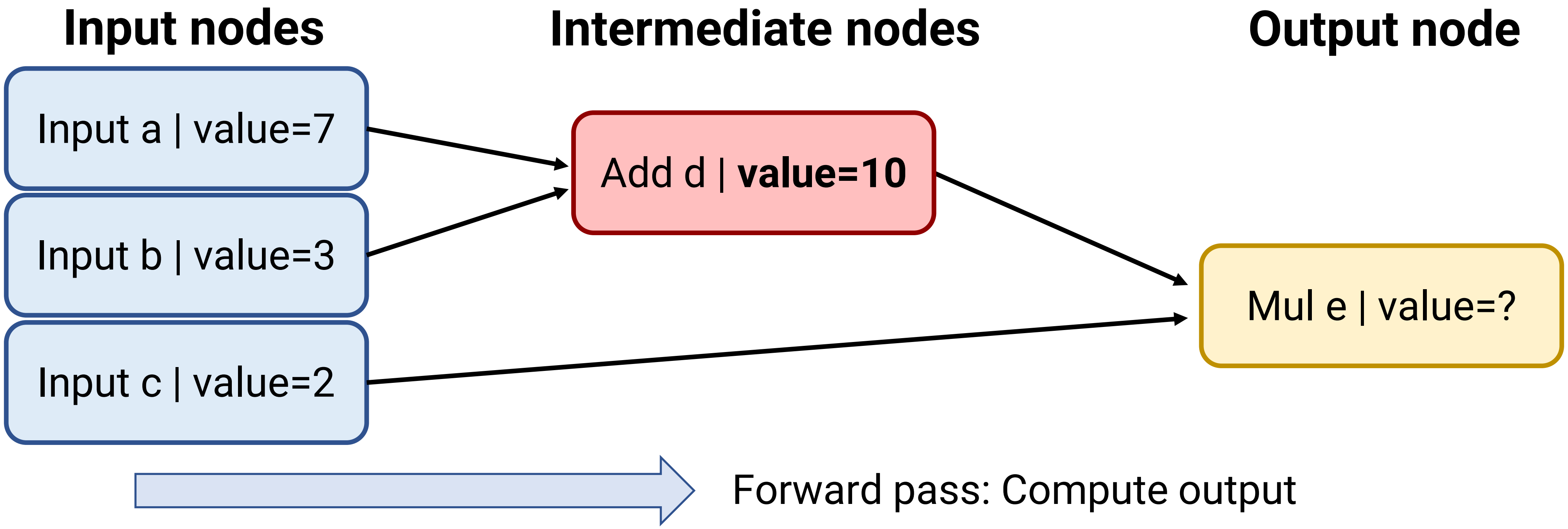
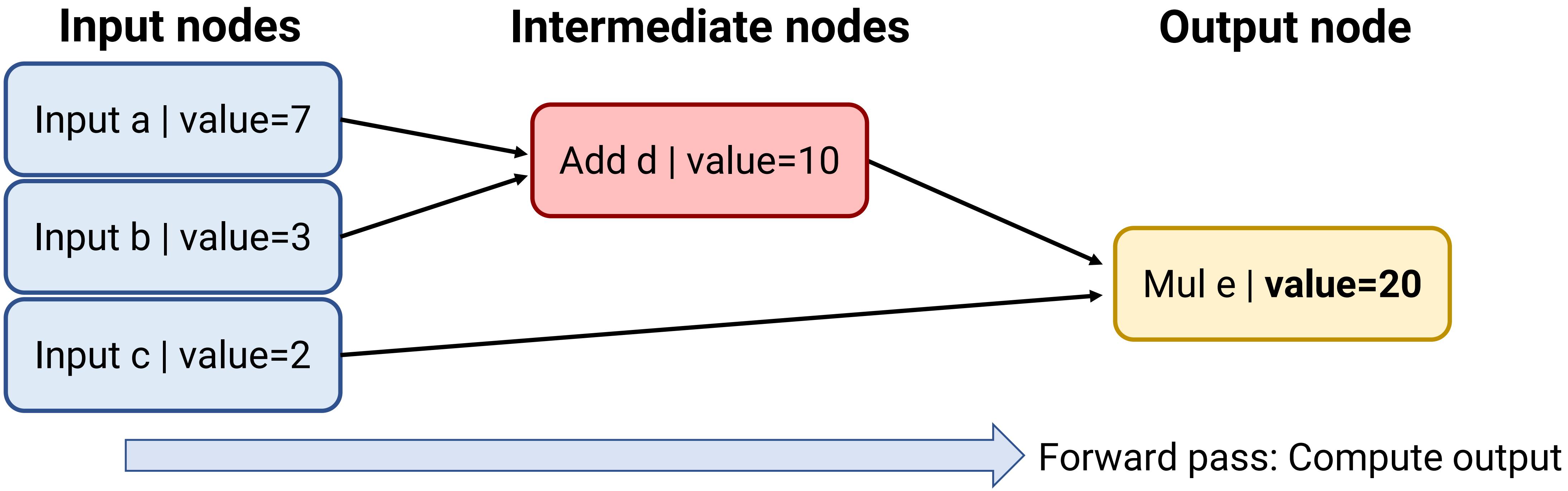


Forward pass: Compute output

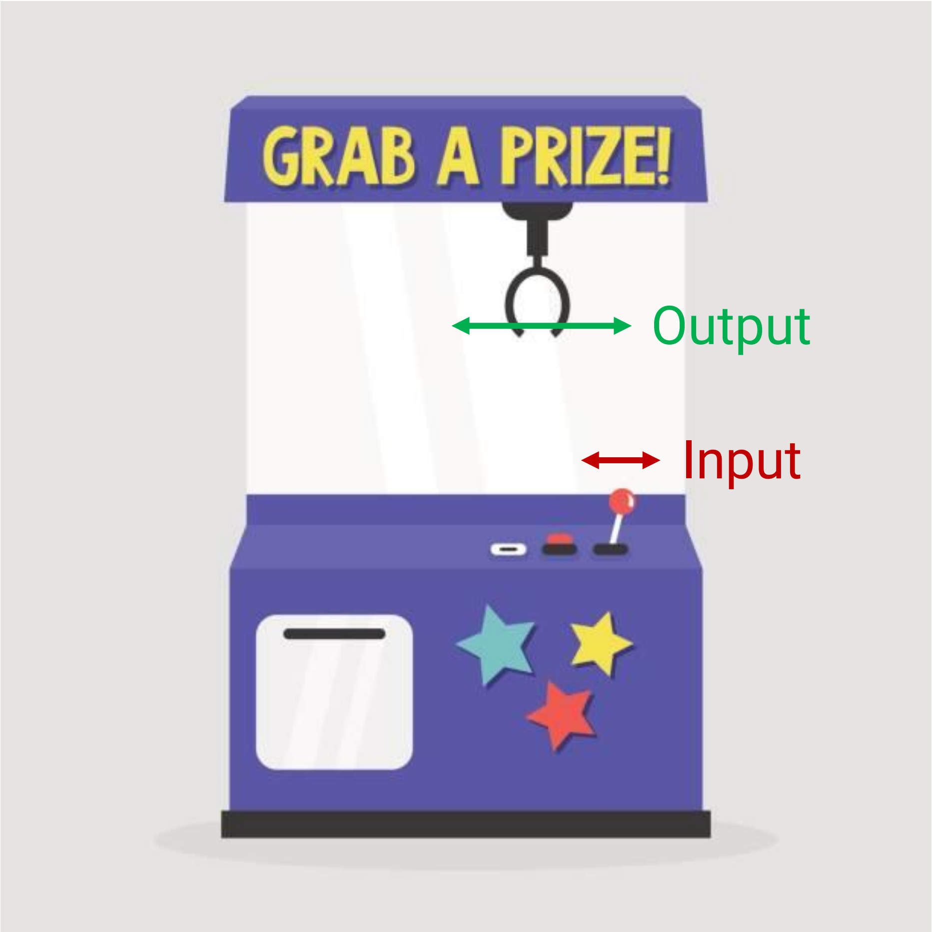
Computation Graph

Computation graph for ***(a + b) * c*** when *a=7, b=3, c=2*



Gradient checking

- **Numerical gradients**: A simpler but less efficient way to compute gradients
- What does *∂y/∂x* mean?
  - If I change *x* by epsilon, by what proportion of epsilon does *y* change?
- We can just compute this for every input node!
- Pro: Easy to implement, useful to check correctness
- Con: Slow—requires O(#inputs) function evaluations



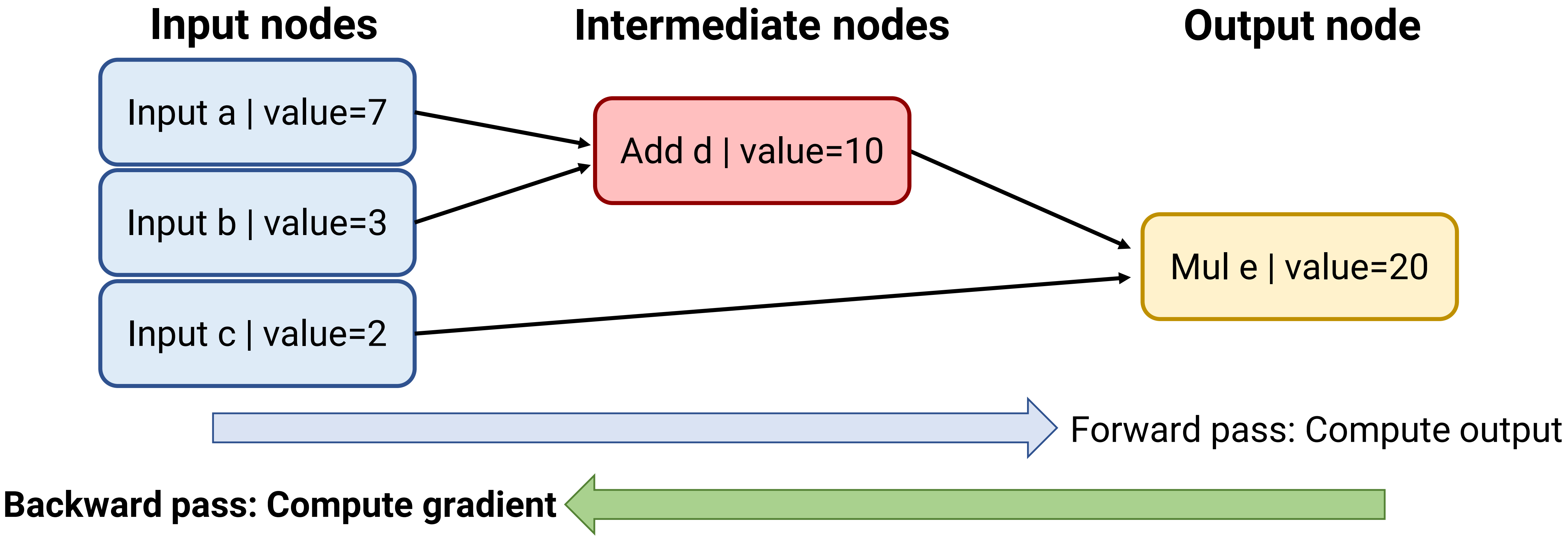
Let’s implement!

Today's Plan

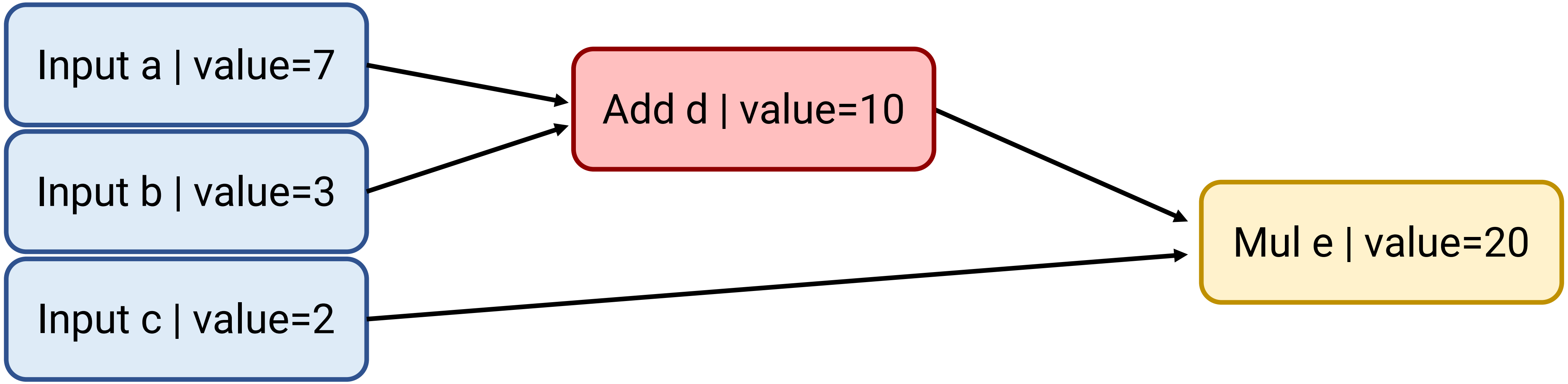
- The computation graph
- Backpropagation on trees
- Backpropagation on DAGs

Computation Graph

Computation graph for ***(a + b) * c*** when *a=7, b=3, c=2*

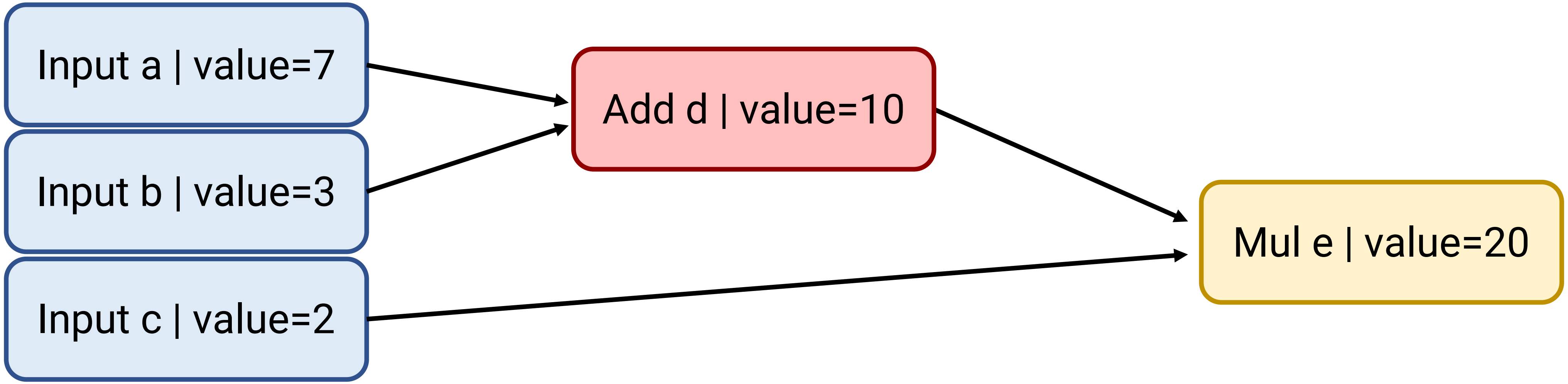


Backpropagation on trees



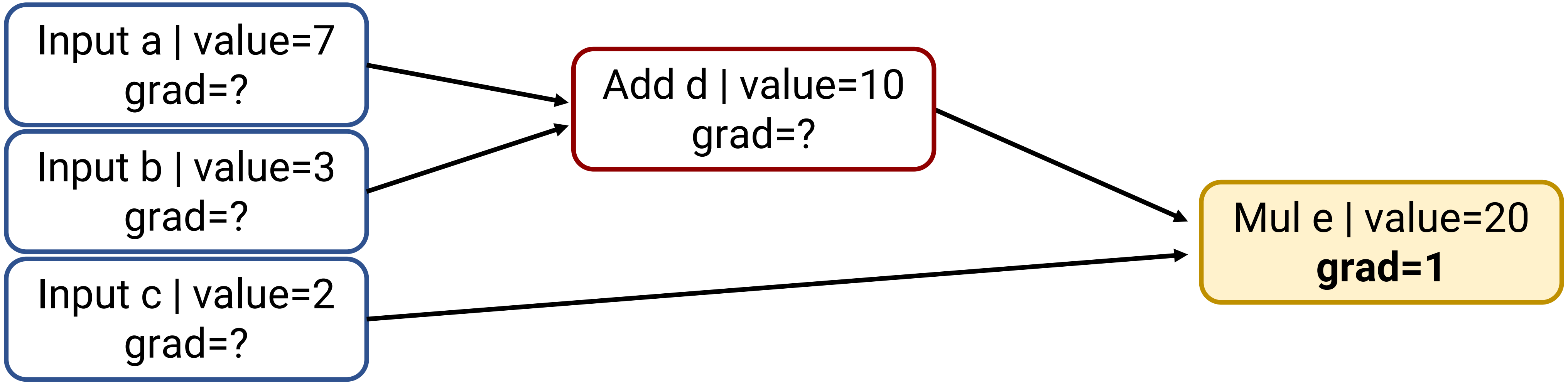
- For now: assume that the computation graph is a tree
  - Each node is only used in a single computation
  - Root of tree is output
  - Leaves of tree are inputs
- Idea: Recursively compute ***∂(output)/∂(node)*** **for each node**, starting at output

Backpropagation on trees



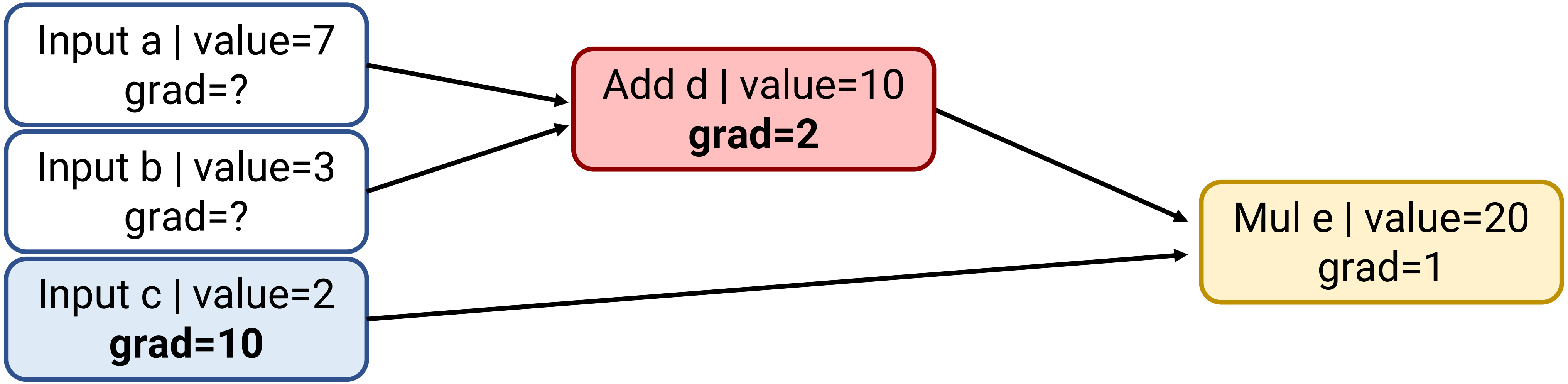
- Goal: Compute gradient *[∂e/∂a, ∂e/∂b, ∂e/∂c]*

Backpropagation on trees



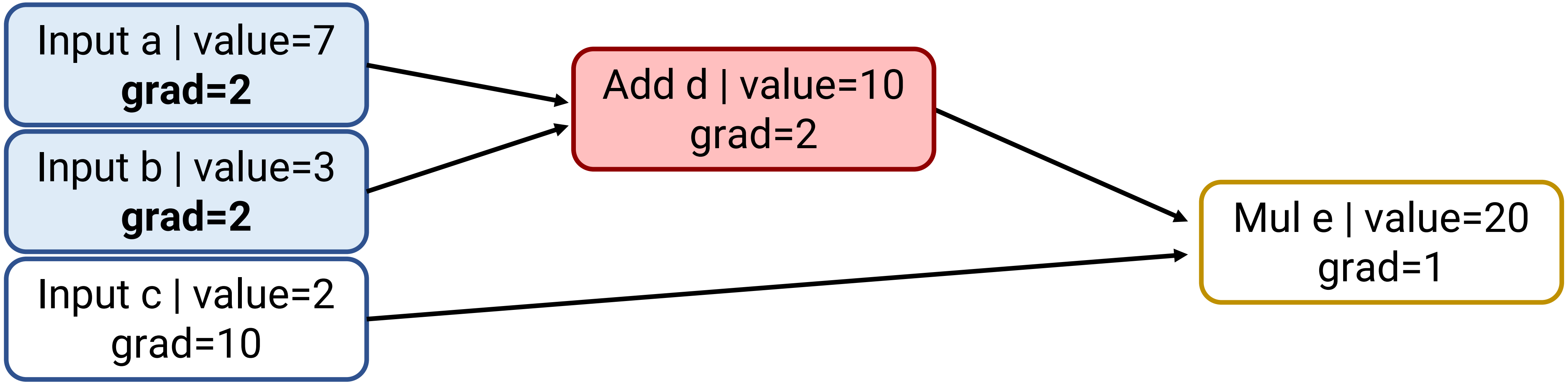
- Goal: Compute gradient $[\partial e/\partial a, \partial e/\partial b, \partial e/\partial c]$
- Step 1: Base case: $\partial e/\partial e = 1$

Backpropagation on trees



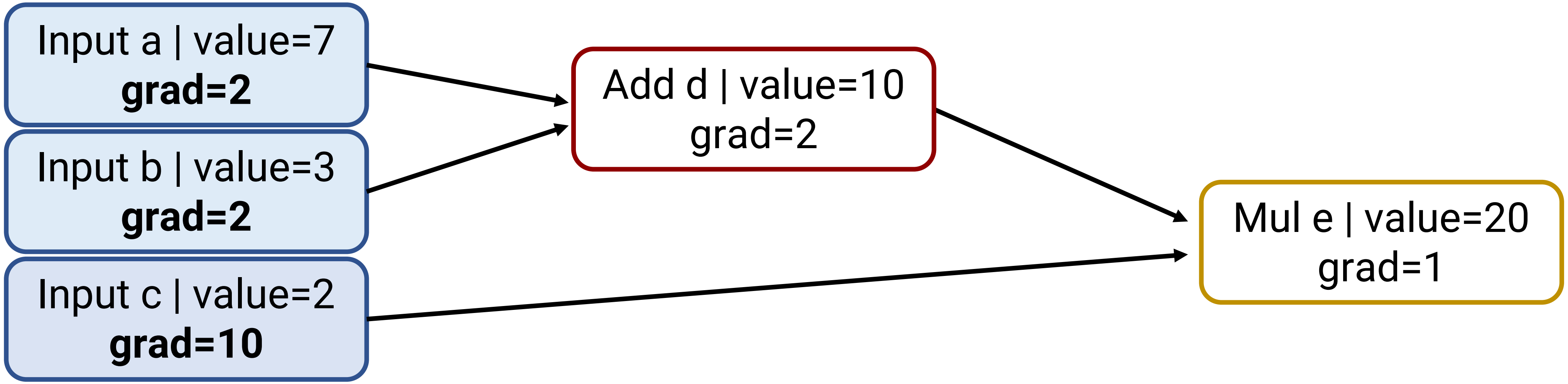
- Goal: Compute gradient *[∂e/∂a, ∂e/∂b, ∂e/∂c]*
- Step 2: How does **Mul** (node e) "distribute" gradient to its children?
  - *∂(x*y)/∂x = y*
  - Chain Rule: *∂(out)/∂x* = *∂(out)/ ∂(x*y)* * *∂(x*y)/∂x* = *∂(out)/ ∂(x*y)* * *y*
  - General rule: Child gets parent's gradient * value of other child

Backpropagation on trees



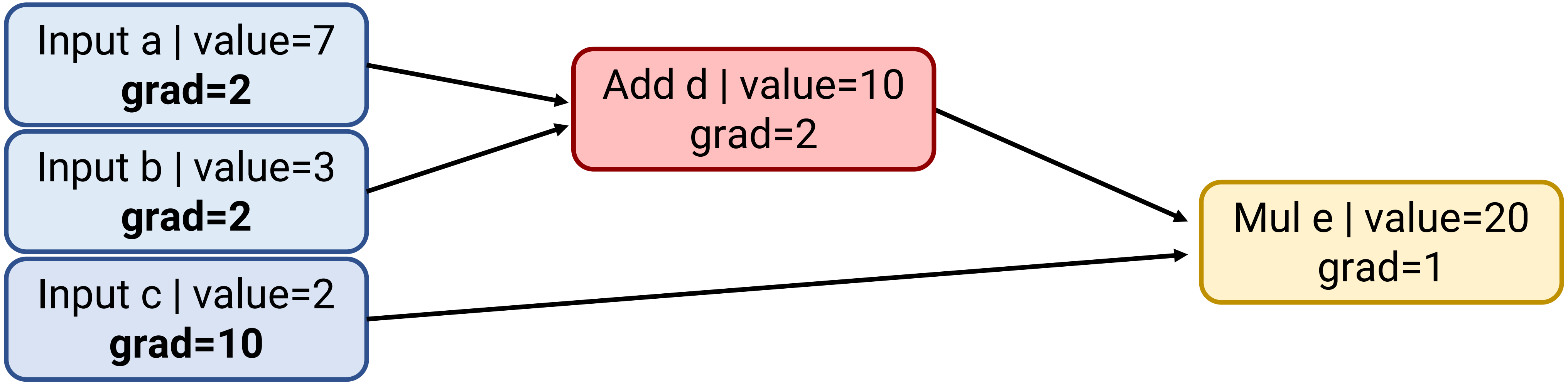
- Goal: Compute gradient *[∂e/∂a, ∂e/∂b, ∂e/∂c]*
- Step 3: How does **Add** (node d) "distribute" gradient to its children?
  - *∂(x+y)/∂x = 1*
  - Chain Rule: *∂(out)/∂x* = *∂(out)/ ∂(x+y)* * *∂(x+y)/∂x* = *∂(out)/ ∂(x+y)* * *1*
  - General rule: Child gets parent's gradient * 1

Backpropagation on trees



- Goal: Compute gradient *[∂e/∂a, ∂e/∂b, ∂e/∂c]*
- Step 4: Leaf nodes
  - Don’t need to do anything

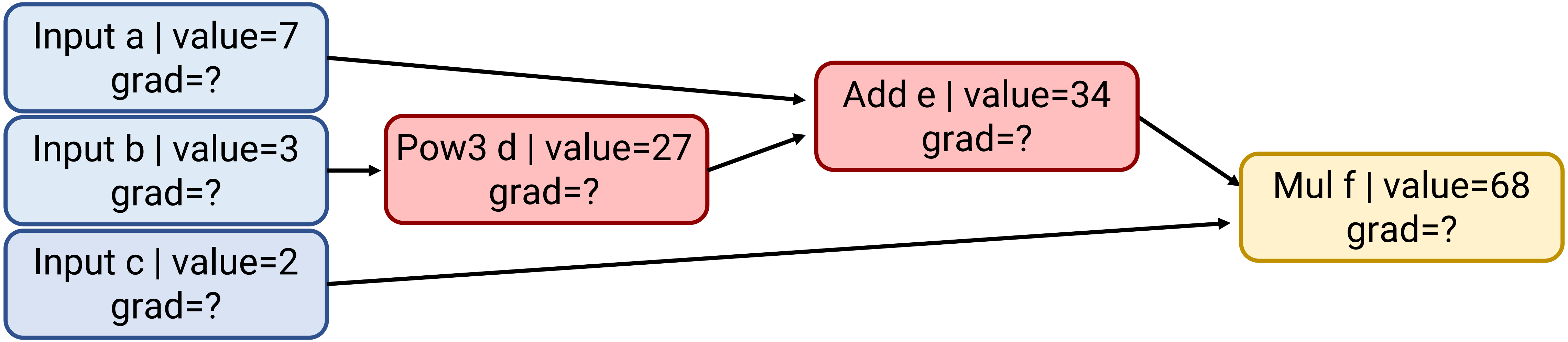
Backpropagation on trees



- Goal: Compute gradient *[∂e/∂a, ∂e/∂b, ∂e/∂c]*
- Overall Recipe
  - Do forward pass
  - Start at root and recurse over children
  - Each node knows how to take gradient of itself with respect to each child
  - By Chain Rule, child.grad = parent.grad * *∂(parent)/∂(child)*

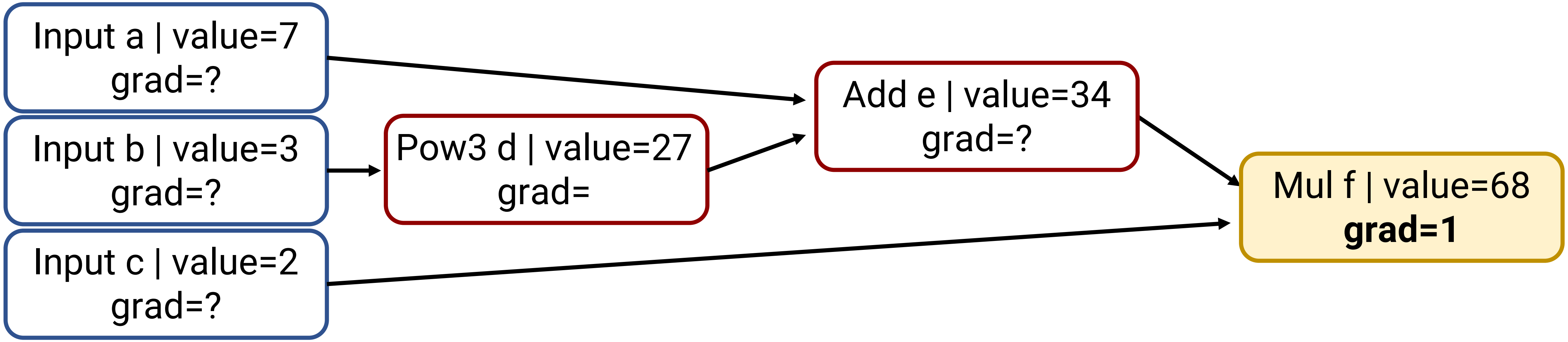
Let's implement!

PowerNode



New function: $(a + b^3) * c$ where $a=7, b=3, c=2$

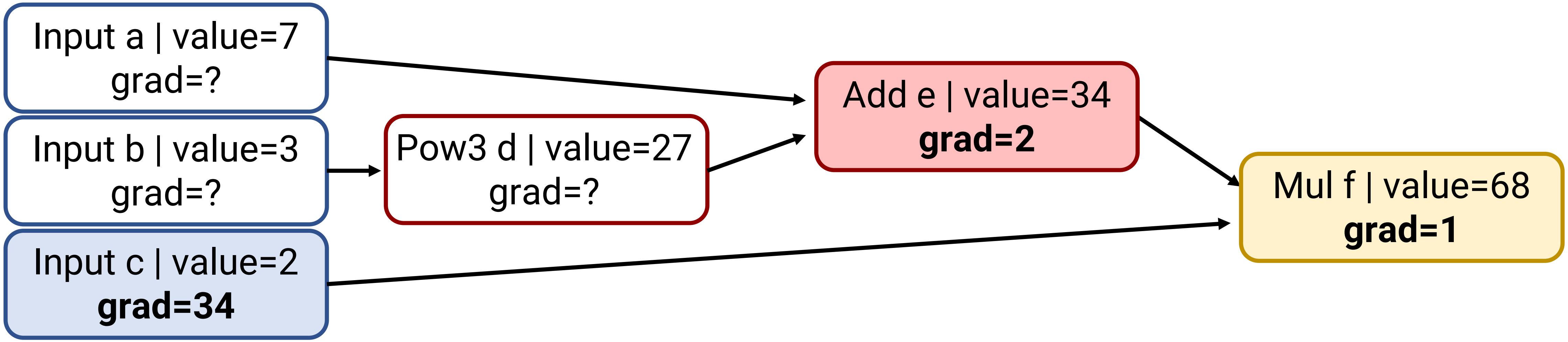
PowerNode



<u>New function: *$(a + b^3) * c$* where *$a=7$, $b=3$, $c=2$*</u>

- Goal: Compute gradient *$[\partial f/\partial a, \partial f/\partial b, \partial f/\partial c]$*
- Step 1: Base case: *$\partial f/\partial f = 1$*

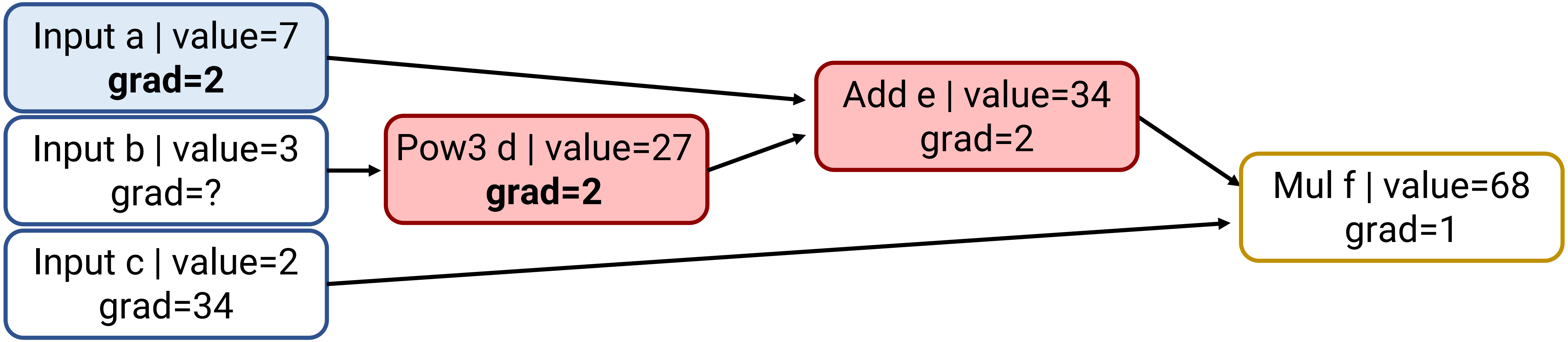
PowerNode



<u>New function: *(a + b³) * c* where *a=7, b=3, c=2*</u>

- Goal: Compute gradient *[∂f/∂a, ∂f/∂b, ∂f/∂c]*
- Step 2: Distribute **Mul** (node f) gradient to children

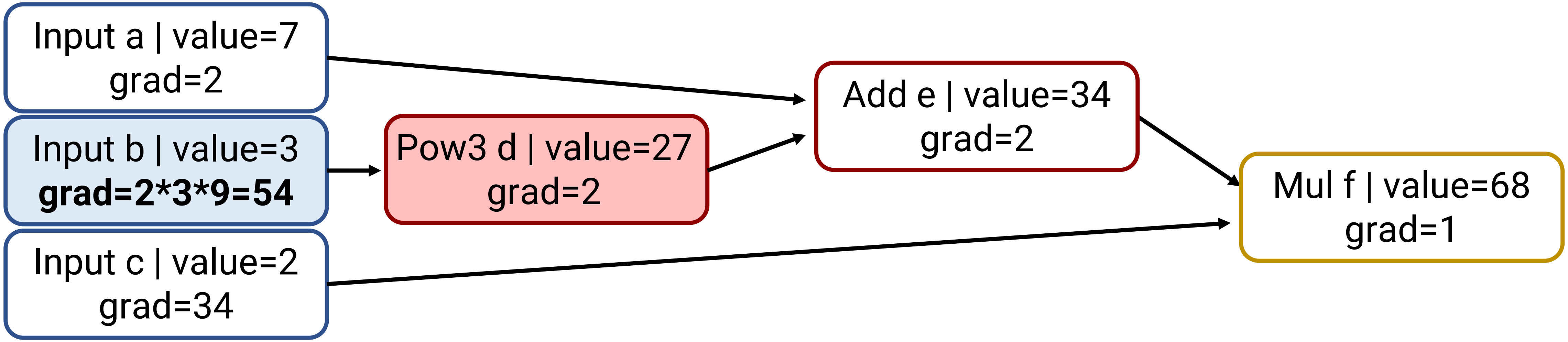
PowerNode



New function: *$(a + b^3) * c$* where *$a=7$, $b=3$, $c=2$*

- Goal: Compute gradient *[∂f/∂a, ∂f/∂b, ∂f/∂c]*
- Step 3: Distribute **Add** (node e) gradient to children

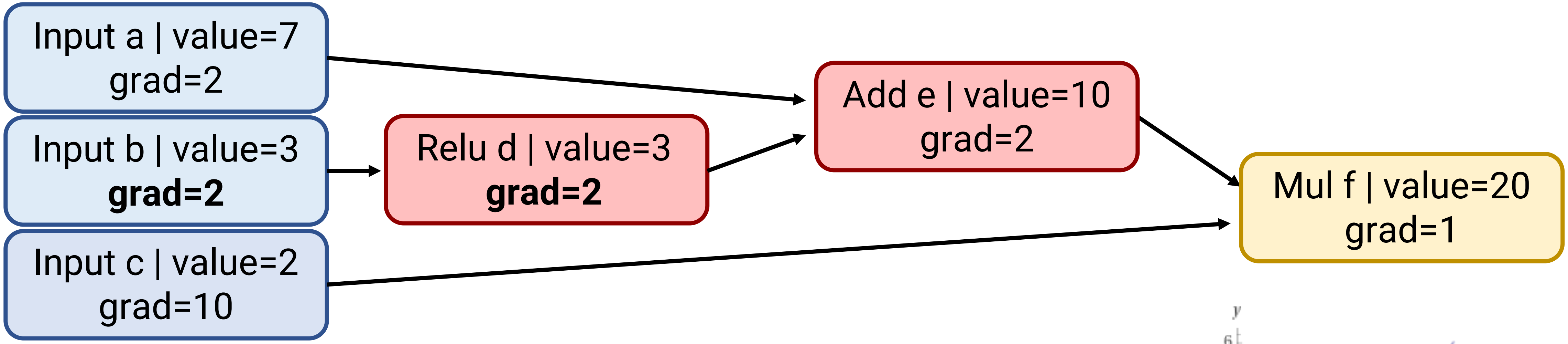
PowerNode



New function: $(a + b^3) * c$ where $a=7, b=3, c=2$

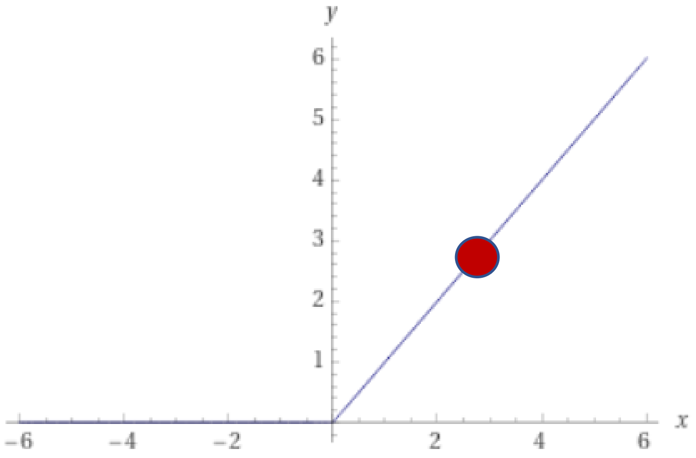
- Goal: Compute gradient $[\partial f/\partial a, \partial f/\partial b, \partial f/\partial c]$
- Step 4: Distribute **Pow3** (node d) gradient to children
  - $\partial(x^p)/\partial x = p * x^{p-1}$
  - By Chain Rule: Child gets parent's gradient * $p * child^{p-1}$

ReluNode



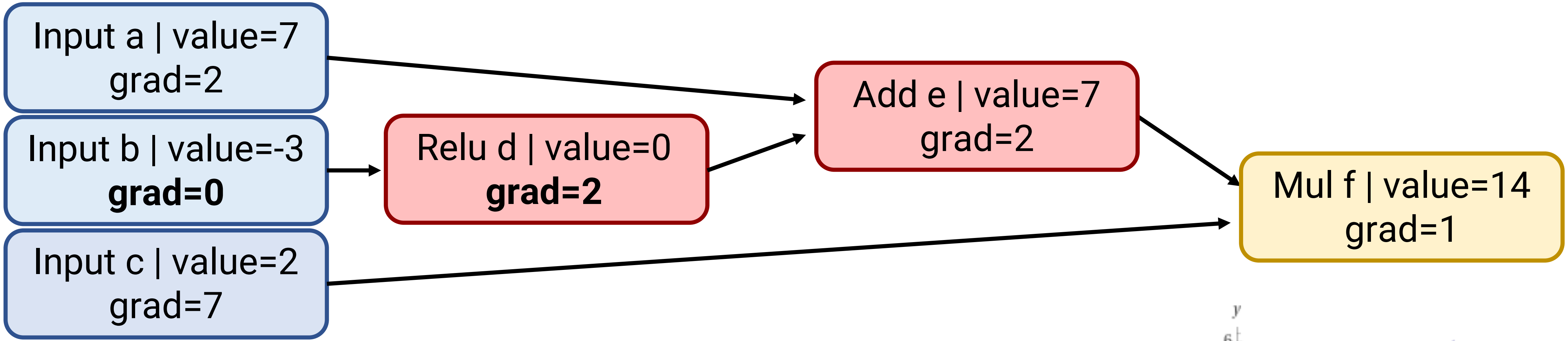
<u>New function: *(a + Relu(b)) * c* where *a=7, b=3, c=2*</u>

- Steps 1-3 are the same
- Step 4: Relu
  - *∂(Relu(x))/∂x = 1 if x > 0, 0 if x ≤ 0*
  - **If child > 0, child.grad = parent.grad * 1**
  - If child ≤ 0, child.grad = 0



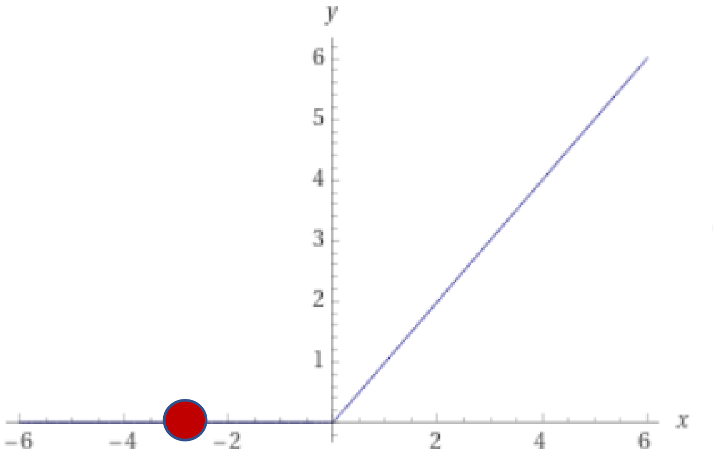
$$\mathrm{ReLU}(z) = \max(z, 0)$$

ReluNode



<u>New function: *(a + Relu(b)) * c* where *a=7*, ***b=-3***, *c=2*</u>

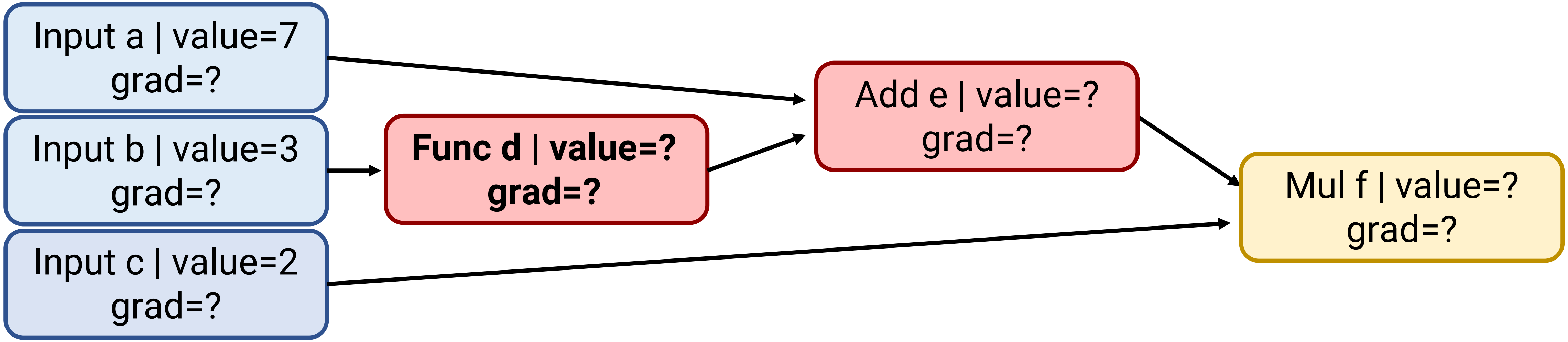
- Steps 1-3 are the same
- Step 4: Relu
  - *∂(Relu(x))/∂x = 1 if x > 0, 0 if x ≤ 0*
  - If child > 0, child.grad = parent.grad * 1
  - **If child ≤ 0, child.grad = 0**



$$\mathrm{ReLU}(z) = \max(z, 0)$$

Let’s implement!

Generic Unary Function



<u>New function: *(a + Func(b)) * c* where *a=7, b=3, c=2*</u>

- Steps 1-3 are the same
- Step 4: Func (generic function)
  - child.grad = parent.grad * *∂(Func(child))/∂child*

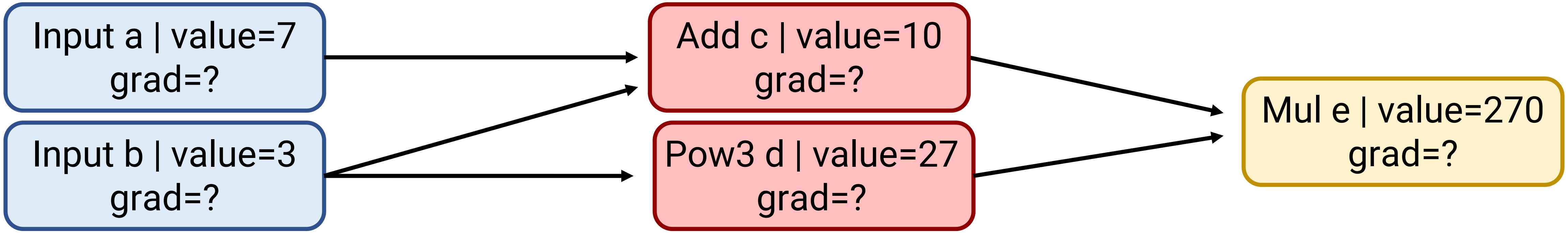
Announcements

- Project proposals due today @ 11:59pm
  - Submit as a group on Gradescope, one submission per group
- HW2 released, due Thursday, October 5
- Midterm exam Tuesday, October 10
  - In-class, 80 minutes
  - Allowed one double-sided 8.5x11 sheet of notes
  - I highly recommend writing this yourself (good for memory)

Today’s Plan

- The computation graph
- Backpropagation on trees
- Backpropagation on DAGs

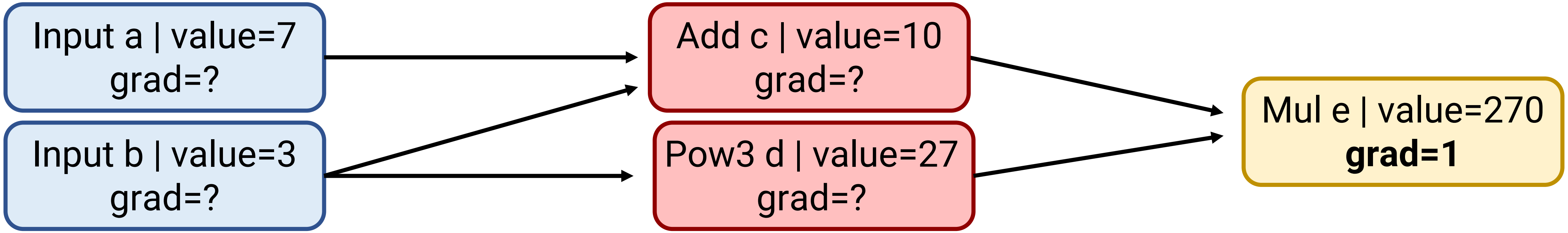
DAG Computation Graphs



New function: *(a + **b**) * **b**^{3}* where *a=7, b=3*

- This is no longer a tree!
- Still a directed acyclic graph
- Let's see why our previous algorithm fails

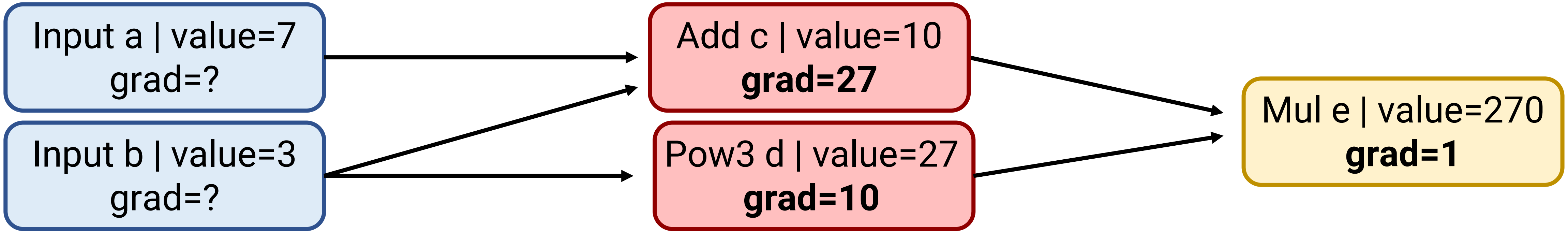
DAG Computation Graphs



<u>New function: *(a +* ***b****)* * $\boldsymbol{b}^3$ where *a=7, b=3*</u>

- Goal: Compute gradient *[∂e/∂a, ∂e/∂b]*
- Step 1: Base case: *∂e/∂e = 1*

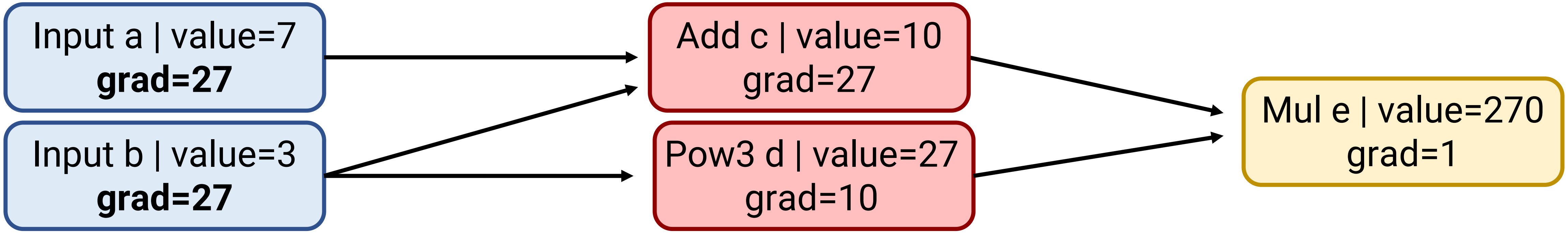
DAG Computation Graphs



New function: *(a +* ***b****)* * ***b****³* where *a=7, b=3*

- Goal: Compute gradient *[∂f/∂a, ∂f/∂b]*
- Step 2: Distribute **Mul** (node e) gradient to children

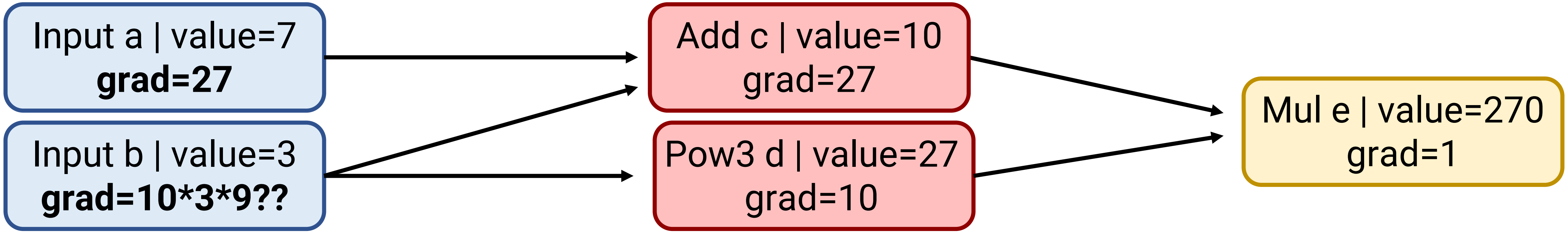
DAG Computation Graphs



New function: *(a +* ***b****)* *** *b***^{3} where *a=7, b=3*

- Goal: Compute gradient *[∂e/∂a, ∂e/∂b]*
- Step 3: Distribute **Add** (node c) gradient to children

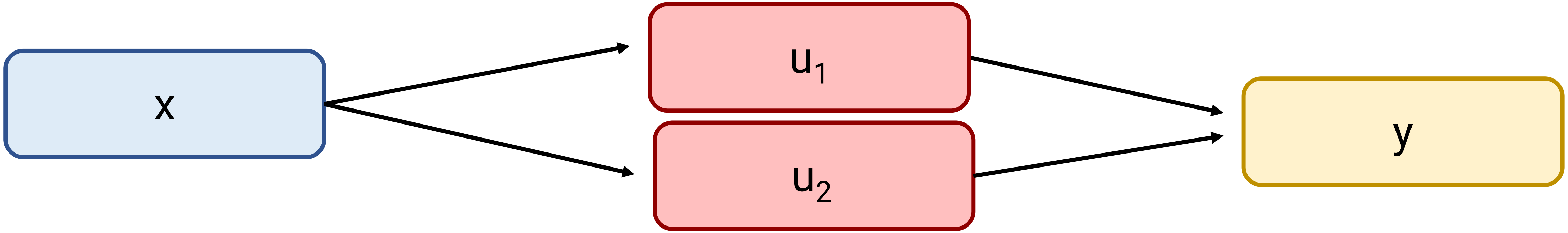
DAG Computation Graphs



<u>New function: *(a +* ***b****)* * $\boldsymbol{b}^3$ where *a=7, b=3*</u>

- Goal: Compute gradient *[∂e/∂a, ∂e/∂b]*
- Step 4: Distribute **Pow3** (node d) gradient to child
  - By Chain Rule: Child gets parent's gradient * *p * child*$^{p-1}$
- Problem: We have overwritten the gradient from b to Add (node c)!

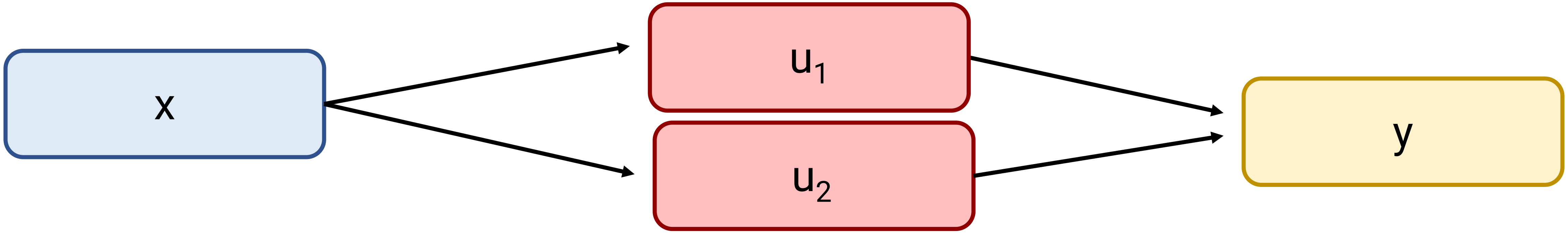
Multivariate chain rule



- Minimal example
  - u_1 and u_2 depend on variable x (i.e., x has two parents)
  - y depends on both u_1 and u_2
- By Multivariate Chain Rule: $\partial y/\partial x = \boxed{\partial y/\partial u_1 * \partial u_1/\partial x + \partial y/\partial u_2 * \partial u_2/\partial x}$
  - Changing x by epsilon changes each parent a bit; Effects add for small epsilon
  - Generalization of multiplying derivatives is matrix multiplication of Jacobians

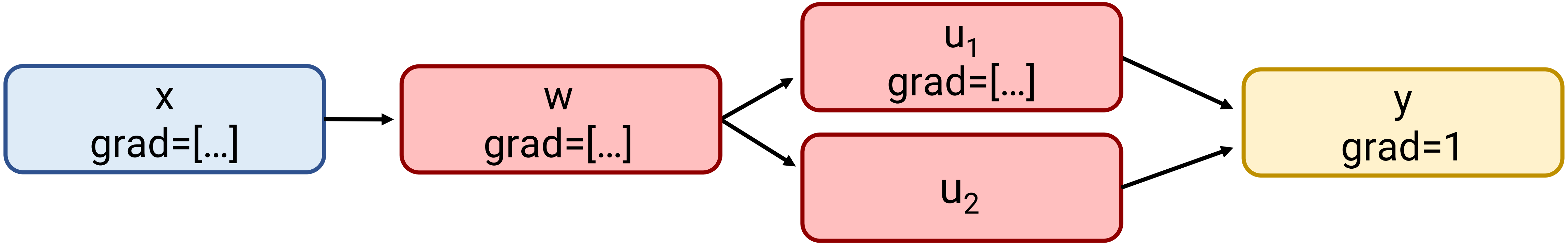
Dot product of
$[\partial y/\partial u_1, \partial y/\partial u_2]$ &
$[\partial u_1/\partial x, \partial u_2/\partial x]$

Multivariate chain rule



- Minimal example
  - u_1 and u_2 depend on variable x (i.e., x has two parents)
  - y depends on both u_1 and u_2
- By Multivariate Chain Rule: $\partial y/\partial x = \partial y/\partial u_1 * \partial u_1/\partial x + \partial y/\partial u_2 * \partial u_2/\partial x$
- node.grad = **sum over parents** of parent.grad * *∂(parent)/∂(child)*
- At each parent node, run child.grad **+=** parent.grad * *∂(parent)/∂(child)*

What order of traversal?

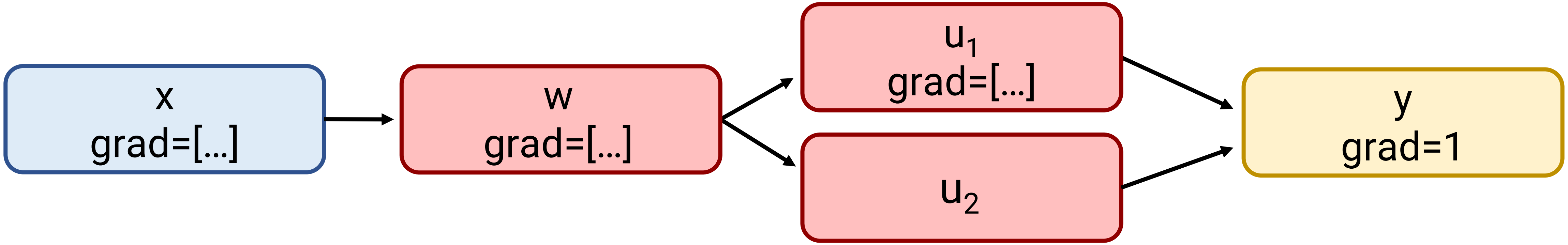


<u>Current code:</u>

- y.backward()
  - u1.backward()
    - **w.backward()**
      - **x.backward()**
  - u2.backward()
    - **w.backward()**
      - **x.backward()**

- Going recursively double-counts
  - First call to w.backward() makes final x.grad too large
- Solution: **Topological sort the nodes**
  - Iterate in reverse order, starting from output
  - Ensures that we process each node after all of its parents

What order of traversal?

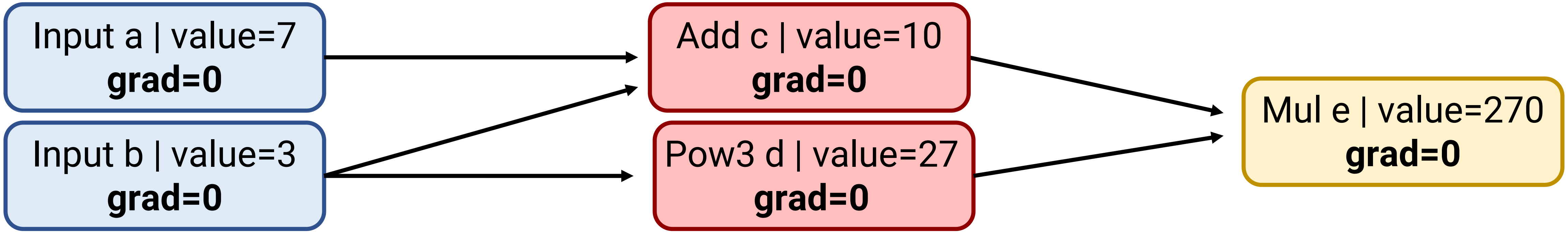


Better code:

- topo_order = [x, w, u_1, u_2, y]
- Iterate in reverse order:
  - y.backward()
  - u2.backward()
  - u1.backward()
  - w.backward()
  - x.backward()

- Going recursively double-counts
  - First call to w.backward() makes final x.grad too large
- Solution: **Topological sort the nodes**
  - Iterate in reverse order, starting from output
  - Ensures that we process each node after all of its parents

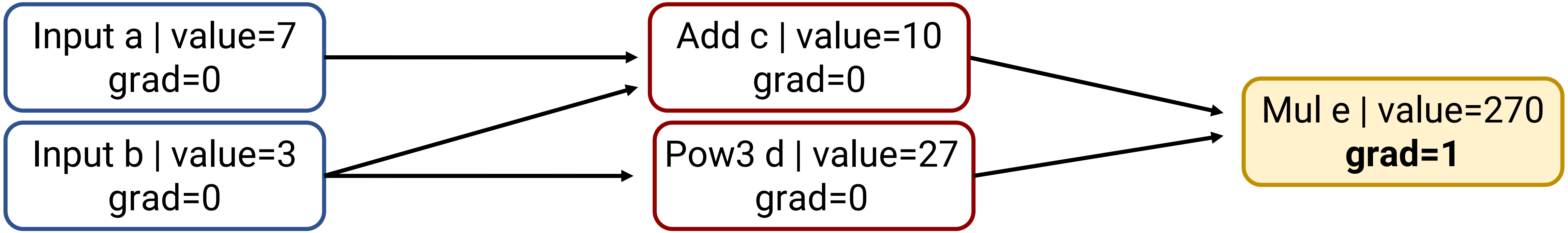
Backpropagation on DAGs



New function: *(a +* ***b****)* * ***b***^{3} where *a=7, b=3*

- Goal: Compute gradient *[∂e/∂a, ∂e/∂b]*
- Topological sort: [a, b, c, d, e]
- Step 0: Initialize all gradients to 0

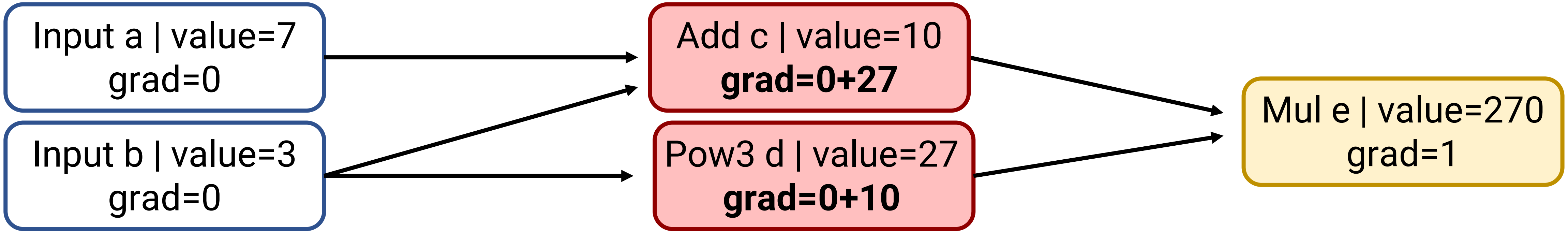
Backpropagation on DAGs



New function: *(a +* ***b****)* * $\boldsymbol{b}^3$ where *a=7, b=3*

- Goal: Compute gradient *[∂e/∂a, ∂e/∂b]*
- Topological sort: [a, b, c, d, e]
- Step 1: Base case: *∂e/∂e = 1*

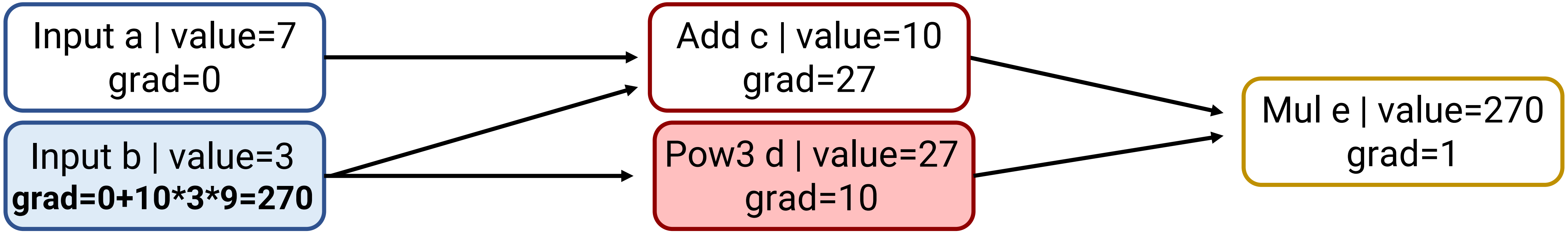
Backpropagation on DAGs



New function: *(a +* ***b****)* *** ***b***^{3} where *a=7, b=3*

- Goal: Compute gradient *[∂e/∂a, ∂e/∂b]*
- Topological sort, reversed: [e, d, c, b, a]
- Step 2: Propagate **Mul** node e to children

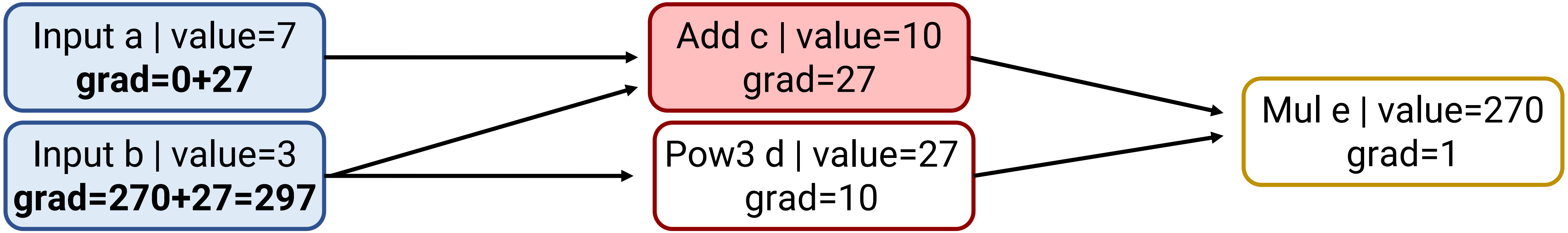
Backpropagation on DAGs



<u>New function: *(a + **b**) * **b**³* where *a=7, b=3*</u>

- Goal: Compute gradient *[∂e/∂a, ∂e/∂b]*
- Topological sort, reversed: [e, d, c, b, a]
- Step 3: Propagate **Pow3** node d to child

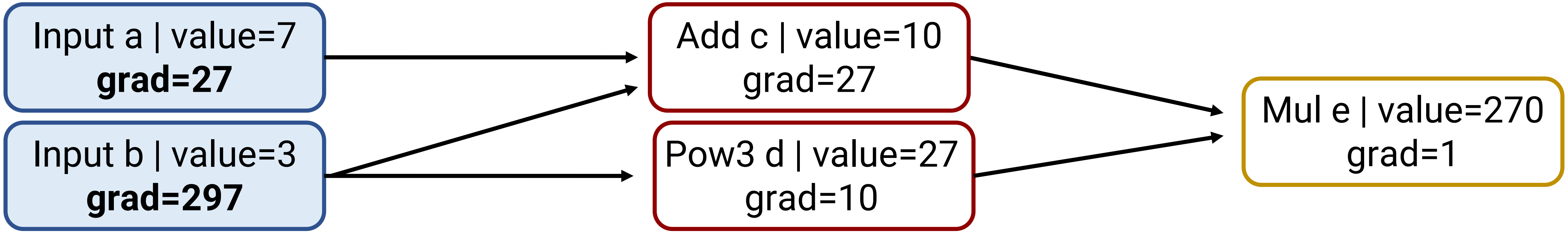
Backpropagation on DAGs



New function: *(a +* ***b****)* *** b^3** where *a=7, b=3*

- Goal: Compute gradient *[∂e/∂a, ∂e/∂b]*
- Topological sort, reversed: [e, d, c, b, a]
- Step 4: Propagate **Add** node c to children

Backpropagation on DAGs

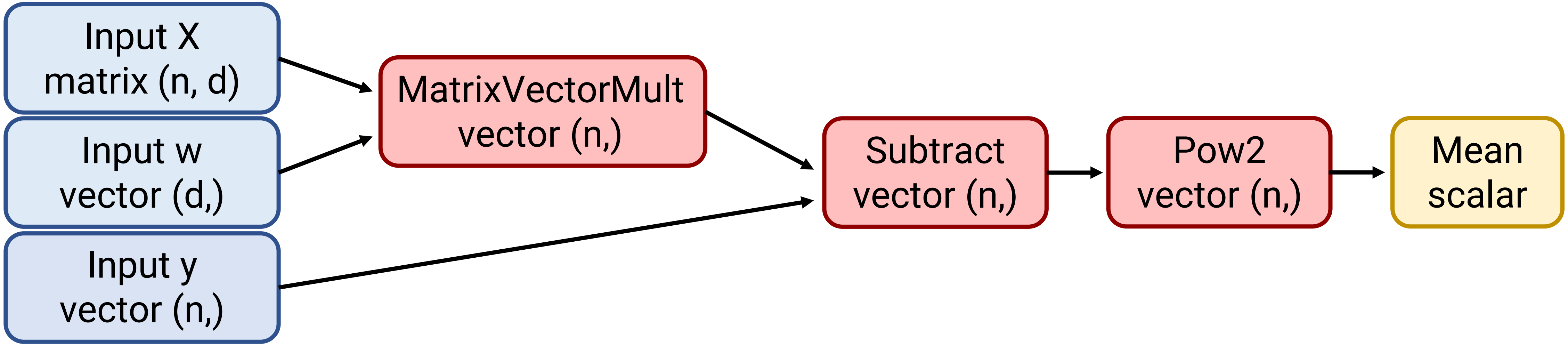


New function: *(a + **b**) * **b**^{3}* where *a=7, b=3*

- Goal: Compute gradient *[∂e/∂a, ∂e/∂b]*
- Topological sort, reversed: [e, d, c, b, a]
- Step 5, 6: a.backward(), b.backward() do nothing

Let’s implement!

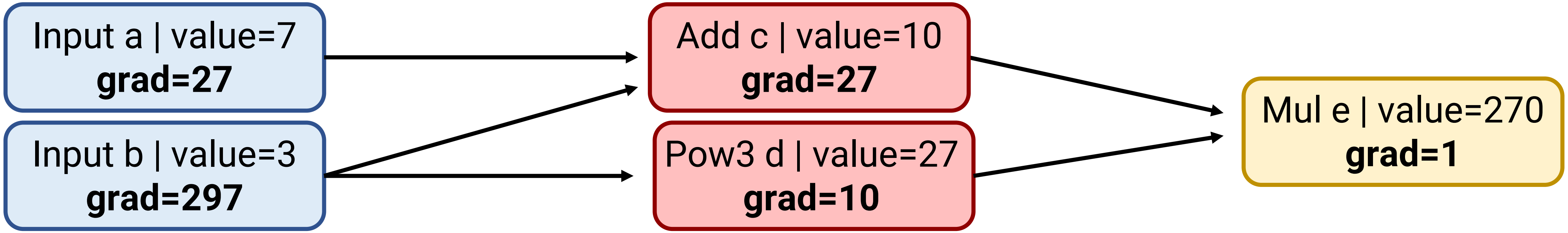
Backpropagation for vectors and matrices



Computation graph for mean($(Xw - y)^2$), i.e. Linear Regression

- Basically the same, but each node can be a vector or matrix!
  - Each node.grad stores ∇_{node}output
  - Parents know how to update ∇_{child}output based on ∇_{parent}output

Conclusion



- Backpropagation computes gradient of output with respect to all nodes in computation graph
  - Forward pass: Compute values of all nodes
  - Backward pass: Iterate through nodes in reverse order,
    At each parent node, run child.grad **+=** parent.grad * *∂(parent)/∂(child)*
- Big picture: Makes it easy to run gradient descent on arbitrary computation graphs
  - Easy to try new architectures for neural networks